May 2002

On whiteness formulas

Introduction

After the recognition of whiteness being a special color attribute o objects much work has been invested in defining whiteness as a unique number that takes into account the colorimetric perception of the object under observation and assessment. Quantification of perceived whiteness has been intimately related to lightness levels and the absence of any hue. White pigments show however in general a yellow shade originating from impurities (mainly Iron ions in different oxidation stages) that has led to the terms "near white" and "off-white"; these colors are really perceived under comparison with a "preferred" white though and this is a result of the chromatic adaptation of the eye.
As a consequence first attempt to quantify whiteness have been related to the quantification of yellowness, lower yellowness values are considered as showing higher whiteness. Almost all pigments are quantified in this way, remarkably also pulps and natural raw fibers that are treated by a chemical bleaching process to lower yellowness values.
The colorimetric compensation of yellowness by adding a blue (or violet) dye is known as "bluing" and was quite widespread in the textile area, specially hand washing where chemical bleaching does not apply. The addition of a dye leads irreversible to a loss of lightness, the object appears grayish or duller as compared with the not treated one, the compensation of the yellow hue is interpreted by the eye as an increase in whiteness though. The situation turns more complicated with the fact that a slight blue shade is also interpreted by the eye as an increase in whiteness and leads to a wider definition of whiteness as having a finite amount of hue and therefore emphasizing the importance of a preferred white.
The use of fluorescence as a method to increment both lightness and blue hue introduces a formidable task to elaborate a formula that takes into account a colorimetric compensation based on additive and sub-tractive color mixing.
The original idea of characterizing whiteness through a unique number is valid today only after the definition of a preferred white has been defined that is highly dependent of the cultural group of the observer and the application of the white object. From a technical point of view formulas based on colorimetric quantities give more information since they are based on psychometric quantities; nowadays efforts are invested in the definition of a whiteness subspace that localizes the perceived color within the color solid, though the definition of preferred white axis numbers can be then transformed into unique perceived whiteness that are strictly part of a color appearance whiteness model.
Literally hundreds of whiteness formulas exist and have been applied in the past, only a selected number of them a presented and discussed in the next sections.

Primitive formulas

The first attempts of describing whiteness are based on just lightness, yellowness or blueness. The formula:

$$W = Y$$

tries to quantify whiteness just as related to lightness, normally as a relative quantity to a preferred white defined by a Magnesium oxide or Barium sulfate tablet, using the CIE function $y(\lambda)$ to describe the luminance factor under a given observer and illuminant. This is purely a luminance value and does not report if the observed object is bluish or yellowish (or by the same token having any other hue). The formula

$$W = B$$

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tries to relate whiteness to a blue reflectance defined either by the CIE function *z(λ)* or to some *ad hoc* defined one as for example the paper brightness function *B(λ)*; the relation to a MgO or $BaSO_4$ preferred white is implicitly contained in the definition of the function B. It is clear that the formula gives always positive numbers regardless of the real color of the observed substrate, furthermore numbers are not corrected by the relative amount of absorbed yellow light and as such it does not take into account the action of bluing techniques. The latter point can be corrected by using some yellowness index that takes into account the relative amounts of blue and yellow in the reflected light; first attempts to describe yellowness were based on the use of special band pass filters as for example:

$$W = \frac{R_{700} - R_{450}}{R_{700}}$$

where R_λ is the reflectance value at the wavelength λ. Further examples are the Stephansen formula:

$$WI_{Stephansen} = 2 \cdot R_{430} - R_{670}$$

and the Harrison formula:

$$WI_{Harrison} = 100 - R_{670} + R_{430}$$

Worth mentioning is the whiteness measured by the Leukometer of VEB Carl Zeiss, Jena, GDR:

$$WI_{Leukometer} = 2 \cdot R_{459} - R_{614}$$

The use of not standardized band pass filters contributed to a loss of popularity of this type of formulas, specially after the introduction of filter colorimeters that are based on the filters G (green), B (blue) and A (amber) that are related to the CIE *y(λ)*, *z(λ)* and the red-portion of the *x(λ)* color-matching functions weighted by a CIE standard illuminant (normally of type A). In general following relationships apply:

Amber filter	R_x	A
Green filter	R_Y	G
Blue filter	R_z	B

$$X = a \cdot R_X + b \cdot R_Z$$
$$Y = R_Y$$
$$Z = c \cdot R_Z$$

$$R_X = \frac{1}{a} \cdot X - \frac{b}{a \cdot c} \cdot Z$$
$$R_Y = Y$$
$$R_Z = \frac{1}{c} \cdot Z$$

where

Observer	Illuminant	a	b	c
2°	A	1.044623	0.053849	0.355824
	C	0.783185	0.197520	1.182246
	D_{65}	0.770180	0.180251	1.088814
10°	A	1.05719	0.05417	0.35202
	C	0.77718	0.19566	1.16144
	D_{65}	0.768417	0.179707	1.073241

Within this formalism yellowness formulas take the form:

$$W = \frac{A - B}{G}$$

and subsequently the Stephansen formula is:

$$WI_{Stephansen} = (2 \cdot R_Z) - R_X$$

and the Harrison formula is:

$$WI_{Harrison} = R_Z - R_X + 100$$

The formula of Taube:

$$W_{Taube} = G - 4 \cdot (G - B)$$

was develop by subtracting the amount of yellowness (second term) from the index of lightness and can also be expressed as:

$$W_{Taube} = 4 \cdot B - 3 \cdot G$$

Closely related is also the whiteness index of the ASTM:

$$WI = 3.388 \cdot Z - 3 \cdot Y$$

Formulas based on a uniform color system (UCS)

It was quite early recognized that yellowness formulas or those based on the relative differences of blue and yellow light were not sufficient to describe whiteness, specially of those objects whitened through bluing techniques. The importance of lightness was recognized as an important contribution to whiteness perception as with the Hunter formula:

$$W_{Hunter} = L - 3 \cdot b$$

where *L* and *b* are Hunter coordinates defined as:

$$L = 100 \cdot \sqrt{\frac{Y}{Y_n}}$$

$$a = 175 \cdot \sqrt{\frac{0.0102 \cdot X_n}{Y/Y_n}} \cdot \left(\frac{X}{X_n} - \frac{Y}{Y_n} \right)$$

$$b = 70 \cdot \sqrt{\frac{0.00847 \cdot Z_n}{Y/Y_n}} \cdot \left(\frac{Y}{Y_n} - \frac{Z}{Z_n} \right)$$

and (X_n, Y_n, Z_n) are the coordinates of the achromatic point. The simplicity of the Hunter formula is remarkable and takes clearly into account the importance of having high lightness and neutral blue *b* values.
Close relatives of this formula are the MacAdam formula given by:

$$W_{MacAdam} = \sqrt{Y - k \cdot p_c^2}$$

where p_e is the colorimetric purity and *k* is a constant that depends on the application, and the Judd formula given by:

$$WI_{(Judd,1936)} = \sqrt{Y - 6700 \cdot (\Delta S)^2}$$

where ΔS is the distance between the sample and the preferred white in the Judd's UCS triangle. The factor 6700 is optimized for grading laundry white goods and may assume a different value for other applications.
The closely related formula:

$$WI_{(Coppock)} = 10 \cdot \sqrt{Y - 2 \cdot p_e^2}$$

is due to W.A. Coppock and known as the Chemstrand Whiteness Scale.
Further formulas based on the principle of colorimetric purity are the Vaeck formula:

$$W_{Vaeck} = Y + k \cdot E(u,v)$$

where the equivalent luminescence *E(u,v)* is defined in the MacAdam UCS diagram and its value for a particular (*u,v*) must be looked up in a nomogram and it defines the dominant wavelength of 472 nm as preferred whiteness hue, and the formula of Anders and Daul:

$$W_{Anders-Daul} = 2 \cdot Y + 1520 \cdot \left[arctg\left(\frac{0.5 - x}{y} \right) - arctg\left(\frac{0.5 - x_n}{y_n} \right) \right] - 65$$

where (x_n, y_n) is the coordinate of achromatic point for D_{65}.
The fact that deviations from the neutral blue yellow axis may contribute to perceived whiteness leads to the Hunter-Judd formula:

$$W_{Hunter-Judd} = 1 - \sqrt{\left[30 \cdot \sqrt{(a^2 + b^2)} \right]^2 + \left[\frac{1 - Y}{2} \right]^2}$$

where *a* and *b* are Hunter coordinates as defined above.
In this respect the original Hunter formula can be generalized as:

$$W = 100 - \sqrt{(L_p - L)^2 + (a^2 + b^2)^2}$$

where L_p is the lightness of the preferred white, in case of MgO it assumes the value 100.

Further developments are the first Selling formula:

$$W_{Selling} = 100 - \sqrt{100 \cdot \Delta\left(Y^{1/2} \right)^2 + k \cdot (\Delta s)^2}$$

with

$$\Delta\left(Y^{1/2} \right) = \sqrt{Y_{MgO}} - \sqrt{Y_{sample}}$$

and

$$\Delta s = \sqrt{(\Delta u)^2 + (\Delta v)^2}$$

being the distance between the sample and the preferred white on the MacAdam's UCS diagram and *k* is a constant.
A simplification of the latter is the second Selling formula:

$$W_{Selling} = 100 - \sqrt{100 \cdot (\Delta Y)^2 + k' \cdot (\Delta s)^2}$$

with

$$\Delta Y = Y_{MgO} - Y_{sample}$$

and Δs defined as above and k' is a constant with the typical value of 9.5 10^6.
The Croes formula is given as:

$$WI_{Croes} = Y - 13.2 \cdot Y \sqrt{(u - u_n)^2 + (v - v_n)^2}$$

where (u,v) are coordinates in MacAdam UCS diagram and (u_n,v_n) are the coordinates of the preferred white.

A last formula worth mentioning is the Friele formula given by:

$$W_{Friele} = \sqrt{A \cdot L^2 - \left(\frac{M}{b}\right)^2 + \left(\frac{S}{c}\right)^2}$$

where (*A,b,c*) are constants and (*L,M,S*) are the length of long, medium and short axis of color discrimination ellipsoid centered on the preferred white. This formula is remarkable since it recognizes fully the importance of deviations from the neutral blue axis as contributions to perceived whiteness, furthermore it attempts to compensate for the different sensitivities from lightness, hue and chroma contributions.

Formulas considering fluorescence

The extensive use of Fluorescent Whitening Agents (FWA) to increase perceived whiteness achieves the compensation of substrate yellowness through an additive color mixing process; a considerable amount of blueness can be introduced without loosing luminance, on the contrary a modest lightness increase results in objects showing dazzling whites.
Depending on their chemical structure, fluorescence produced by FWAs can lead to neutral, or to red- or green-shaded whiteness; the existence of shade preferences is illustrated by the formula:

$$WI_{(C429)} = 100 - \sqrt{\left(\frac{220 \cdot (G - B)}{G + 0.242 \cdot B}\right)^2 + \left(\frac{100 - G}{2}\right)}$$

originally due to Hunter but modified to give a neutral white preference, or the formula

$$WI_{(CDML)} = \frac{L - 3 \cdot b + 10 \cdot \sqrt{Y} - 21 \cdot (Y - Z)}{\sqrt{Y}}$$

that shows a blue white preference.
Due to the additive nature of the process it was readily recognized that linear formulas could be built for measuring perceived whiteness in any of the colorimetric spaces:

$$W = \beta \cdot B + \gamma \cdot G + \alpha \cdot A + k_1$$
$$W = \lambda \cdot L + \nu \cdot b + \mu \cdot a + k_2$$
$$W = \varepsilon \cdot Y + \rho \cdot x + \sigma \cdot y + k_3$$

since they can be regarded as variations of the same theme; this represents also a first rationalization of existing formulas, since most of them can fit in one of the listed formulas, allowing a classification of origins and preferences.

The question of preferred white was however delegated to second importance since FWA manufacturers tried to develop whiteness formulas tailored to the characteristics of their products.
The formula of Stensby:

$$W_{Stensby} = L + 3 \cdot a - 3 \cdot b$$

derives from the Hunter formula and shows clearly a preference for redder whites, while the formula of Berger:

$$W_{Berger} = Y + a \cdot Z - b \cdot X$$

with

	a	b
2° observer	3.400	3.895
10° observer	3.448	3.904

shows a preference for green whites as well the formula of Croes:

$$WI_{Croes} = R_Y + R_Z - R_X$$

Much of the development of linear whiteness formulas was done by Ganz, who formulated a general formula as:

$$W_{Ganz} = Y + P \cdot (x_0 - x) + Q \cdot (y_0 - y)$$

where the values of the parameters determine the hue preference as seen from the table:

	hue preference		
	red	neutral	green
P	-800	+800	+1700
Q	+3000	+1700	+900

where (x_0,y_0) is the coordinate of the achromatic point for the D_{65} illuminant.
At this stage it was clearly recognized the importance of the amount of UV acting onto the sample, daylight conditions were chosen as reference for work on whiteness determination and modern formulas are strictly valid for D_{65}.

Modern formulas: general linear forms

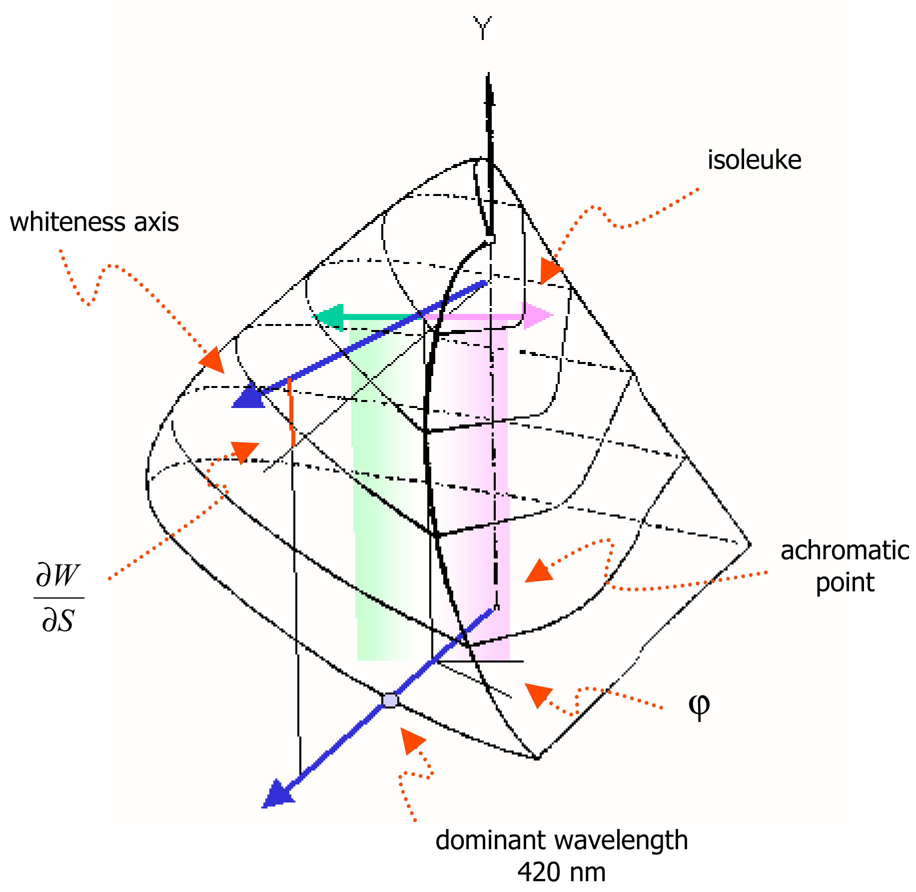
Starting point is the Roesch color solid as depicted in the figure, under following conditions:

- Illuminant D_{65}
- Dominant wavelength for neutral whites 470 nm

The plane with the said dominant wavelength describes (blue) colors with the same spectral purity at different levels of color saturation *S*, the colors perceived as white will lie within a limited region on this plane; perpendicular to this plane are the ones corresponding to hues.
Curves with same whiteness are called isoleukai, these are defined by the whiteness formula (regardless of its general form), it must be remarked that an isoleuke contains the same values for whiteness *W* but different values for hue (or shade deviation).

The isoleukai for Y=100 and λ_d= 470 nm consist of fairly parallel and equally spaced flat curves that can be approximated by straight lines; this is the case for example for the formulas of Berger and Stensby but the lines have different slope because of their different preference for greenish or reddish whites and is controlled by the angle φ in the figure.
To set up the whiteness formula following parameters must be determined numerically:

- The gain of whiteness with increasing saturation $\partial W/\partial S$
- The impact of lightness on whiteness $\partial W/\partial Y$
- The impact of the hue on whiteness $\partial W/\partial H$



In general the following relationship holds:

$$\left(\frac{\partial W}{\partial H}\right) = -\left(\frac{\partial W}{\partial S}\right)\cdot \tan(\varphi)$$

and each whiteness point is characterized by the slope of the isoleukai:

$$\omega = \frac{\left(\partial W / \partial S\right)}{\left(\partial W / \partial Y\right)}$$

and their angle φ with respect to the line with λ_d= 470 nm:

$$\varphi = 45 - arctg \frac{\left(\partial W / \partial S\right) + \left(\partial W / \partial H\right)}{\left(\partial W / \partial S\right) - \left(\partial W / \partial H\right)}$$

It must be remarked that the numerical value of (*∂W/∂S)* sets the extension of the whiteness scale and is closely related to the amount of UV present in the illuminant; this is a direct consequence of the presence of fluorescence resulting from excitation of the FWA and inherent to the nature of the formula.

With these definitions following linear formula can be written down:

$$W_{Ganz} = D \cdot Y + P \cdot (x_0 - x) + Q \cdot (y_0 - y)$$

where

$$D = \frac{\partial W}{\partial Y}$$

$$P = -\left(\frac{\partial W}{\partial S}\right) \cdot \left(\frac{\cos(\varphi + \eta)}{\cos(\varphi)}\right)$$

$$Q = -\left(\frac{\partial W}{\partial S}\right) \cdot \left(\frac{\sin(\varphi + \eta)}{\cos(\varphi)}\right)$$

where η is the angle between the line with λ_d= 470 nm and the x axis, and $(x_0.y_0)$ is the achromatic point for D_{65}.

The next step is the evaluation of the shade deviation or tint by the formula:

$$T_{Ganz-Griesser} = m \cdot (x_0 - x) + n \cdot (y_0 - y)$$

where

$$m = \frac{-\cos(\alpha)}{BW}$$

and

$$n = \frac{\sin(\alpha)}{BW}$$

and α is the angle of the perpendicular to the line with λ_d= 470 nm and the bandwidth BW is a constant related to the sensitivity of the eye to distinguish different shades of white.

The Ganz and Ganz-Griesser formulas

The formulas are expressed by:

$$W_{Ganz} = Y + P \cdot (x_0 - x) + Q \cdot (y_0 - y)$$

$$T_{Ganz-Griesser} = m \cdot (x_0 - x) - n \cdot (y_0 - y)$$

and describe whiteness and shade deviation (tint) for daylight D_{65}. The formula parameters adopt following numerical values:

- D= *∂W/∂Y* = 1
- *∂W/∂S*= 4000
- dominant wavelength is 470 nm (*η*= 48.18154° and α=41.81852°)
- φ= 15° (light preference for greenish whites)
- BW= 0.0008

Under these conditions following parameters are calculated:

- *P* = 1868.322
- *Q* = 3695.690
- *m* = 931.576
- *n* = -833.467
- $(x_0.y_0)$ = (0.313795,0.330972)

The numerical scale sets up following threshold values:

- threshold for undistinguishable whiteness sample pairs: 5 Ganz whiteness points
- threshold for undistinguishable shade deviation in sample pairs: 0.5 Ganz-Griesser points

The instrument for conducting the measurements must be equipped with a device to regulate the amount of UV falling onto the samples, this amount must be set (and maintained) to an amount similar to that encountered in daylight in order to obtain reliable whiteness data. Some problems arise because small unavoidable physical differences among instruments result in large discrepancies in measured whiteness numbers; for this reason the Ganz-Griesser formulas are applied with instrument-specific parameters calculated with the aid of proper calibrated samples. The formulas are expressed as:

$$W_{Ganz} = Y + P \cdot x + Q \cdot y + C$$

$$T_{Ganz-Griesser} = m \cdot x + n \cdot y + k$$

where the values of (*P,Q,C*) and (*m,n,k*) must be determined by a fitting procedure using a set of calibration samples; thus they have not an universal character and apply only for the specific calibrated instrument; this procedure leads to satisfactory results when inter-instrumental comparison (specially shade deviation values) is mandatory, though to the price of non-transferable parameters.

The CIE formulas

The formulas standardized by the CIE are expressed by:

$$W_{CIE} = Y + 800 \cdot (x_0 - x) + 1700 \cdot (y_0 - y)$$

$$T_{CIE} = 900 \cdot (x_0 - x) - 650 \cdot (y_0 - y)$$

and describe whiteness and shade deviation (tint factor) for daylight D_{65} and 10° observer, for the 2° observer the tint factor is given by:

$$T_{CIE} = 1000 \cdot (x_0 - x) - 650 \cdot (y_0 - y)$$

The formula parameters are based on following conditions:

- D= $\partial W/\partial Y$ = 1
- $\partial W/\partial S$= 1800.36
- dominant wavelength is 470 nm (η = 48.18154° and α = 41.81852°)
- φ = 16.6173° (light preference for greenish whites)
- BW = 0.000901
- (x_0.y_0) = (0.3127 , 0.3290) for 10° observer and (0.3127, 0.329) for 2° observer

The numerical scale sets up following threshold values:

- threshold for undistinguishable whiteness sample pairs: approx. 2.3 CIE whiteness points
- threshold for undistinguishable shade deviation in sample pairs: approx. 0.2 CIE shade points

As with the Ganz formula, the instrument for conducting the measurements must be equipped with a device to regulate the amount of UV falling onto the samples and this amount must be set (and maintained) to an amount similar to that encountered in daylight in order to obtain reliable whiteness data.
The CIE equations are object of a norm issued by the CIE and adopted by many institutions like ISO, Tappi, AATCC, DIN, ASTM, etc.
Strictly speaking the CIE formulas are valid only for illuminant D_{65} and for UV amounts similar to daylight, however some institutions allow the use of the CIE formulas in conjunction with illuminants other than D_{65}. As shown above the value of $\partial W/\partial S$ determines the scaling of the whiteness values and it is closely related to the amount of fluorescence excited from the FWA; there has been no study about the behavior of the isoleukai for other illuminants. Recently the ISO has extended the CIE formulas to be used in conjunction with the illuminant C (introducing the term “indoor whiteness”), on the grounds that although the amount of UV differs notably from that of daylight, the coordinates of the achromatic point do not differ much. While probably the isoleukai are not too distorted compared with those of daylight, the assumption remains to be proved true.
In a later work Ganz gave the CIE equations in CIE-L*a*b* space as:

$$W_{CIE-L*a*b*} = 2.41 \cdot L^* - 4.45 \cdot b^* \cdot [1 - 0.0090 \cdot (L^* - 96)] - 141.4$$

and $T_{CIE-l*a*b*} = -1.58 \cdot a^* - 0.38 \cdot b^*$

Note the independence of whiteness values of the value of a*.

These formulas correspond to a generalization of the originally postulated linear formulas into the (L*,a*,b*) color space .

Performance of linear whiteness formulas

One must not forget that linear formulas were developed for fluorescent whites, they perform fairly well for medium to high whiteness levels, but the linear approximation starts breaking down for very high whiteness values or for medium to strong shaded samples. For samples with low content of fluorescence or especially for just bleached materials they do

not give reliable data and one should switch to a yellowness formula to obtain proper assessment.
Another problem arising from the linear form is that the formulas are "open", any sample, even a colored one will show certain degree of whiteness; proper assessment requires assistance form the human observer. While limits for whiteness have been postulated, for example:

- samples are white if -20 < W_{Ganz} < (8*Y-490)
- samples are white if 40 < W_{CIE} < (5*Y-280)

the validity of these limits is highly dependent on the observer and they fail specially with heavy shaded samples.
Recently Uchida has proposed corrective terms to the CIE formula as follows:

if 40 < W_{CIE}< 5*Y-275

$$W = W_{CIE} - 2 \cdot (T_{CIE})^2$$

if W_{CIE} > 5*Y-275

$$W = W_{CIE} - 2 \cdot (P_W)^2$$

where

$$P_w = (5 \cdot Y - 275) - \{800 \cdot [U + V \cdot (100 - Y) - x]^{0.82} + 1700 \cdot [R + T \cdot (100 - Y) - y]^{0.82}\}$$

and

	2° observer	10° observer
U	0.2761	0.2742
V	0.00117	0.00127
R	0.2727	0.2762
T	0.0018	0.00176

Conclusion

Literally hundreds of whiteness formulas have been published or applied in different fields over the last 70 years and under a variety of conditions. Certainly the field of whiteness has evolved also during all those years, imposing new additional challenges to the developed formulas, the instrumental data has however stayed back and failed to provide enough reliable data to conduct quantitative and conclusive studies.
Whiteness perception, although occupying a small amount of the total color solid, is still a psychochromatic phenomena attached to three quantities as any other color, this is a hint that a proper description must be based on three quantities that describe fully the perceived whiteness.
Still the quest for the "most beautiful white" remains open, but it is a recognized fact that its definition depends on cultural background of the observer group and application of the object, a specialization of whiteness formulas based on absolute principles seems possible and represents truly the goal of the next developments in this area.